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EXCELLENCE PROGRAM - SYJC (SCIENCE), 2019-2020

SYNOPSIS

MATHEMATICS & STATISTICS – PART 2

DIFFERENTIATION [6 MARKS FOR H.S.C.]

DEFINITION:

If $y = f(x)$ for all real values of x, then $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, if the limit exists, is called the derivative of f w.r.t. x, at x and is denoted by $f'(x)$ or $\frac{dy}{dx}$.

If this limit exists, the function f is said to be differentiable or derivable at x. The process of finding the derivative is called **differentiation**.

Finding the derivative of a given function by using the above definition is referred to as finding its derivative from first principle.

Note: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

LIST OF FORMULAE:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec u)x = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \quad ; \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$$

$$\frac{d}{dx}\left(a^x\right) = a^x \log a$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

CHAIN RULE (DERIVATIVE OF COMPOSITE FUNCTIONS):

THEORY I: If y is a differentiable function of u and u is a differentiable function of x, then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Proof : If increment in x is δx, let δu and δy be the corresponding increments in u and y respectively.

As $\delta x \to 0; \delta u \to 0$ $\quad\therefore \delta y \to 0$

Now $\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}$ $\quad\left[\because \delta x \to 0 \therefore \delta x \neq 0 \quad \because \delta u \to 0 \therefore \delta u \neq 0\right]$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x}=\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}\right)$$

$$\therefore \lim_{\delta x \to 0}\frac{\delta y}{\delta x}=\lim_{\delta x \to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x \to 0}\frac{\delta u}{\delta x} \qquad \ldots\text{(I)}$$

(Limit of a product = Product of limits)

As $\delta x \to 0; \delta u \to 0$

$$\therefore \lim_{\delta x \to 0}\frac{\delta y}{\delta u}=\lim_{\delta u \to 0}\frac{\delta y}{\delta u} \qquad \ldots\text{(II)}$$

From (I) and (II), we get

$$\therefore \lim_{\delta x \to 0}\frac{\delta y}{\delta x}=\lim_{\delta u \to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x \to 0}\frac{\delta u}{\delta x}$$

$\because$ y is a differentiable function of u,

$\lim_{\delta u \to 0}\frac{\delta y}{\delta u}=\frac{dy}{du}$ exists and is finite.

$\because$ u is a differentiable function of x,

$\lim_{\delta x \to 0}\frac{\delta u}{\delta x}=\frac{du}{dx}$ exists and is finite.

$\therefore$ All the limits on R.H.S. exists and are finite

$\Rightarrow$ limits on L.H.S. also exists and is finite.

Hence $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Corollary: Hence evaluate $\frac{d}{dx}[\log f(x)]$

Let y = log f(x) and u = f(x)

$\therefore$ y = log u $\qquad \therefore \frac{dy}{du} = \frac{1}{u}$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= \frac{1}{u} \cdot \frac{du}{dx} \qquad = \frac{1}{f(x)} \frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[\log f(x)] = \frac{f'(x)}{f(x)}$$

Corollaries to Chain Rule:

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\sin u\right) = \cos u \frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos u\right) = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan u\right) = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx}\left(\cot u\right) = -\operatorname{cosec}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}\left(\sec u\right) = \sec u \tan u \frac{du}{dx}$$

$$\frac{d}{dx}\left(\operatorname{cosec} u\right) = -\operatorname{cosec} u \cot u \frac{du}{dx}$$

$$\frac{d}{dx}\left(\log u\right) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\left(a^u\right) = a^u \log a \frac{du}{dx}$$

$$\frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$$

LOGARITHMIC DIFFERENTIATION:

When we want to find the derivative of a function which is expressed as a product of a number of functions or a quotient of functions or is of the form $[f(x)]^{g(x)}$, then it is convenient to find the derivative of the logarithm of the function.

i.e. If $y = [f(x)]^{g(x)}$ then $\frac{dy}{dx} = [f(x)]^{g(x)}\left[\frac{g(x).f'(x)}{f(x)} + g'(x).\log f(x)\right]$

DERIVATIVE OF INVERSE FUNCTIONS:

THEORY II: If $x = f^{-1}(y)$ is a derivable function of y, such that the inverse function $y = f(x)$ is defined, then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \text{ where } \frac{dx}{dy} \neq 0.$$

Proof : If increment in y is δy, let δx be the corresponding increment in x,

$\because$ x is a derivable function of y

$\Rightarrow$ x is a continuous function of y. $\therefore$ As $\delta y \to 0, \delta x \to 0$

Now $\frac{\delta y}{\delta x} = \frac{1}{\frac{\delta x}{\delta y}}$ $\qquad \therefore \lim\limits_{\delta y \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta y \to 0} \frac{1}{\frac{\delta x}{\delta y}}$

$$\therefore \lim_{\delta y \to 0} \frac{\delta y}{\delta x} = \frac{\lim\limits_{\delta y \to 0} 1}{\lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y}} \qquad \ldots(i)$$

As $\delta y \to 0, \quad \delta x \to 0$

$$\therefore \lim_{\delta y \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \qquad \ldots(ii)$$

From (i) and (ii), we get $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\lim\limits_{\delta y \to 0} 1}{\lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y}}$

$\because$ x is a derivable function of y

$\Rightarrow \lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y} = \frac{dx}{dy}$ exists and is finite

$\therefore$ All the limits on R.H.S. exist and are finite.

$\Rightarrow$ L.H.S. limit also exists and is finite.

Hence $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$, where $\dfrac{dx}{dy} \neq 0$

THEORY III: If y = f(x) is a derivable function of y, such that the inverse function

x = f^{-1} (y) is defined, then $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}}$, where $\dfrac{dy}{dx} \neq 0$

Proof : If increment in x is δx, let δy be the corresponding increment in y.

$\because$ y is a derivable function of x $\Rightarrow$ y is a continuous function of x

$\therefore$ As $\delta x \to 0, \delta y \to 0$.

$$\lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \lim_{\delta x \to 0} \frac{1}{\dfrac{\delta y}{\delta x}}$$

$$\therefore \lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \frac{\lim\limits_{\delta x \to 0} 1}{\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}} \qquad \ldots\text{(i)}$$

As $\delta x \to 0$, $\delta y \to 0$.

$$\therefore \lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \lim_{\delta y \to 0} \frac{\delta x}{\delta y} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get,

$$\lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \frac{\lim\limits_{\delta x \to 0} 1}{\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}}$$

$\because$ y is a derivable function of x $\Rightarrow \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$ exists and is finite

$\therefore$ All the limits on R.H.S. exist and are finite.

$\Rightarrow$ Limit on L.H.S. also exists and is finite.

Hence by definition, $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}}$, where $\dfrac{dy}{dx} \neq 0$

Derivatives of Inverse Trigonometric Functions :

1) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

2) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

3) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \varepsilon R.$

4) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, for $x \varepsilon R$

5) $\frac{d(\sec^{-1}x)}{dx} = \frac{1}{\left|x\sqrt{x^2-1}\right|}$ where $|x| > 1.$

6) $\frac{d}{dx}(cosec^{-1}x) = -\frac{1}{\left|x\sqrt{x^2-1}\right|}$, where $|x| > 1.$

DERIVATIVE OF PARAMETRIC FUNCTIONS

THEORY IV: If u and v are differentiable functions of t, such that u is a function of v, then

$$\frac{du}{dv} = \frac{\frac{du}{dt}}{\frac{dv}{dt}}$$

Answer: If increment in t is δt, let δu and δv be the corresponding increments in u and v respectively.

As $\delta t \to 0, \delta u \to 0, \delta v \to 0$

Now, $\frac{\delta u}{\delta v} = \frac{\frac{\delta u}{\delta t}}{\frac{\delta v}{\delta t}}$

$$\therefore \lim_{\delta t \to 0} \frac{\delta u}{\delta v} = \lim_{\delta t \to 0} \frac{\frac{\delta u}{\delta t}}{\frac{\delta v}{\delta t}}$$

$$\therefore \lim_{\delta t \to 0} \frac{\delta u}{\delta v} = \frac{\lim_{\delta t \to 0} \frac{\delta u}{\delta t}}{\lim_{\delta t \to 0} \frac{\delta v}{\delta t}} \qquad \ldots(i)$$

As $\delta t \to 0, \quad \delta v \to 0$

$$\therefore \lim_{\delta t \to 0} \frac{\delta u}{\delta v} = \lim_{\delta v \to 0} \frac{\delta u}{\delta v} \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$\lim_{\delta v \to 0} \frac{\delta u}{\delta v} = \frac{\lim_{\delta t \to 0} \frac{\delta u}{\delta t}}{\lim_{\delta t \to 0} \frac{\delta v}{\delta t}}$$

$\because$ u is a differentiable function of t

$$\Rightarrow \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt} \text{ exists and is finite}$$

$\because$ v is a differentiable function of t

$$\Rightarrow \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} \text{ exists and is finite}$$

$\therefore$ All the limits on R.H.S. exist and are finite.

$\Rightarrow$ limit on L.H.S. also exists and is finite.

Hence $\dfrac{du}{dv} = \dfrac{\frac{du}{dt}}{\frac{dv}{dt}}$